$$r^{\cdot}\sqrt{\nu^{-1}r} - \chi\sqrt{(\nu r)^{-1}}\,\varkappa = \varkappa\sqrt{\frac{1-2\varkappa^2}{1+\varkappa^2}} = \rho(\varkappa)$$

Shedding is possible if $|r^{\cdot}\sqrt{\nu^{-1}r}| \leqslant n(\chi\sqrt{(\nu r)^{-1}})$, where n is found from the condition that the line $y = n - \varkappa\chi\sqrt{(\nu r)^{-1}}$ (shown broken in Fig.5) touches the curve $y = \rho(\varkappa)$ (the heavy line in Fig.5).

4. *The case of a circular initial and final orbit.* We will evaluate the Hamiltonian on the singular surface. By Eq.(3.17) and the first of (3.25), we have

$$H = \lambda_2(f - {}^2/_3 r^{-1}(r^{\cdot})^2),\quad f = \nu r^{-2}(g^2 - 1) < 0$$

On the singular surface, therefore, the Hamiltonian does not vanish. If the control process duration is not fixed, we can add to the conclusion of Sect.3 the fact that the Hamiltonian is continuous at the instants $0, t_p$ Hence the optimal control program does not contain intermediate thrust. It consists of apsidal tangential impulses. If the interorbital transition time is fixed, our analysis shows that the hypothesis of /2/ about the absence of intermediate thrust in problems with variable angular range is equivalent for our present case to continuity of the Hamiltonian at the instant of reaching the given orbit.

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ANALYTIC SOLUTIONS OF THE HAMILTON-JACOBI EQUATION OF AN IRREVERSIBLE SYSTEM IN THE NEIGHBOURHOOD OF A NON-DEGENERATE POTENTIAL ENERGY MAXIMUM*

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The existence of analytic solutions for the Hamilton-Jacobi equations of an irreversible system with two degrees of freedom in the neighbourhood of a non-degenerate maximum of potential energy is investigated. It is shown that these solutions define manifolds in phase space which are filled with trajectories which asymptotically approach an equilibrium position as $t \to \pm\infty$.

Consider a mechanical system with Lagrangian

$$L\colon R^2\{x\} \times R^2\{x^{\cdot}\} \to R,\quad L = T_2 + T_1 - \Pi$$
$$T_2 = {}^1/_2\langle K(x)\,x^{\cdot}, x^{\cdot}\rangle,\quad T_1 = \langle V(x), x^{\cdot}\rangle$$

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where T_2 is the kinetic energy of the system ($K(x)$ is a positive definite matrix, and $\langle\cdot,\cdot\rangle$ denotes the scalar product in R^2), T_1 is a linear form representing velocities which generate gyroscopic forces, and $\Pi(x)$ is the potential energy of the system.

Let us assume that the function Π, the coefficients of the matrix K and the components of the vector field V are analytic functions of the generalized coordinates x. Let the origin $x=0$ be an equilibrium state of the system ($d\Pi(0)=0$) at which the potential energy has a non-degenerate maximum ($\det \partial^2\Pi(0)\,\partial x^2 \neq 0$). We may assume without loss of generality that in the neighbourhood of the origin (E denotes the identity matrix)

$$K(x) = E + K'(x)$$
$$\Pi(x) = -{}^1/_2\langle Dx, x\rangle + \Pi'(x), \quad D = \operatorname{diag}(\omega_1^2, \omega_2^2)$$
$$V(x) = \Omega x + V'(x), \quad \Omega = \begin{Vmatrix} 0 & \omega \\ -\omega & 0 \end{Vmatrix}$$

where the Maclaurin expansions of the coefficients of the matrix K', the function Π' and the vector V' begin with terms that are small to order at least 1, 3, and 2, respectively.

The Hamilton-Jacobi equation corresponding to the Lagrangian L has the form

$$ {}^1/_2\langle K^{-1}(x)(\partial S/\partial x - V(x)), \ \partial S/\partial x - V(x)\rangle + \Pi(x) = h \tag{1}$$

where K^{-1} is the inverse of K, and h is the energy constant. If $S(x)$ is some solution of Eq.(1), the function $-S(x)$ is a solution of the Hamilton-Jacobi equation for the Lagrangian $L = T_2 - T_1 - \Pi$. A particular solution $S\colon R^2\{x\} \to R$ of Eq.(1) defines in the phase space $R^2\{x\} \times R^2\{y\}$ (y is the vector of generalized momenta) an invariant manifold $M = \{(x, y)\colon y = \partial S/\partial x\}$, i.e., an integral curve that has a point in common with M must lie entirely in M. Analysing the equation $y = \partial S/\partial x$, one can obtain various classes of motions of the system.

From the standpoint of local theory, it is interesting to consider the existence of analytic solutions of the Hamilton-Jacobi equation in the neighbourhood of an equlibrium position at an energy level which contains an equilibrium state. We shall therefore assume that $h=0$.

We will seek a solution of Eq.(1) in the neighbourhood of the point $x=0$, as a series

$$S = \sum_{i\geqslant 2} S_i(x), \quad S_i(x) = \sum_{\alpha_1+\alpha_2=i} s_{\alpha_1\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2} \tag{2}$$

Substituting the series into (1), we obtain

$$\langle \partial S_2/\partial x - \Omega x, \ \partial S_2/\partial x - \Omega x\rangle = \langle Dx, x\rangle \tag{3}$$

It can be shown that the quadratic form

$$S_2 = {}^1/_2\langle Ax, x\rangle, \quad A = \frac{1}{a_+}\begin{Vmatrix} |\omega_1| e & \omega a_- \\ \omega a_- & |\omega_2| e \end{Vmatrix}$$
$$e = \sqrt{a_+^2 - \omega^2}, \quad a_\pm = {}^1/_2(|\omega_1| \pm |\omega_2|) \tag{4}$$

satisfies Eq.(3), and the forms S_m ($m = 3, 4, \ldots$) are defined by the equations

$$\langle (A - \Omega)\, x, \ \partial S_m/\partial x\rangle = G_m \tag{5}$$

where G_m are known m-th -order forms. Whether Eq.(5) is solvable depends on the spectrum of the matrix $A - \Omega$. The eigenvalues of this matrix are $k_{1,2} = (e \pm \sqrt{a_+^2 - \omega^2})$. If $a_+ > |\omega|$, the expression $\alpha_1 k_1 + \alpha_2 k_2$ does not vanish for non-negative integers α_j such that $\alpha_1 + \alpha_2 = m$ ($m = 2, 4, \ldots$). Consequently, by Lyapunov's Theorem (/1/, p.66), there exists exactly one form S_m satisfying Eq.(5). Thus, there exists a formal solution of Eq.(1) as a power series (2) with quadratic part (4).

We note that if $a_+ = |\omega|$, the expression $\alpha_1 k_1 + \alpha_2 k_2$ may vanish for even m, and so there need not always exist a solution in the form of a power series.

Theorem. If $a_+ > |\omega|$, Eq.(1) has an analytic solution in the neighbourhood of the equilibrium position $x=0$:

$$S^\pm = {}^1/_2\langle A^\pm x, x\rangle + W^\pm(x); \ A^+ = -A(-\omega)$$

where the matrix A is defined as in (4) and the Maclaurin expansions of the functions $W^\pm$ begin with at least third-order terms.

In the case of a natural system ($T_1 \equiv 0$), this theorem was proved in /2/. In the case $|\omega_1| > |\omega|$, $|\omega_2| > |\omega|$, the existence of a smooth solution of Eq.(1) was proved in /3/.

Corollary. Under the assumptions of the theorem, the invariant manifolds $M^{\pm} = \{(x, y)\ y = \partial S^{\pm}/\partial x\}$ are filled out by phase trajectories asymptotically approaching the equilibrium state as $t \to \mp\infty$.

The corollary is proved by substituting $y = Kx^{\cdot} + V$ into the equation $y = \partial S/\partial x$. The solution $S^+ (S^-)$ generates a system of second-order equations with singular point $x = 0$ of the type of an unstable (stable) node or focus. Lifting the family of asymptotic trajectories $x(t)$ to phase space, we obtain the manifolds $M^{\pm}$.

We note that if $a_+ < |\omega|$ the equilibrium is stable and there are no asymptotic motions. For a linear system with $a_+ = |\omega|$, the manifolds $M^{\pm}$ contain no asymptotic trajectories. In a non-linear system satisfying this condition asymptotic trajectories may exist.

Here is a simple example: $K'(x) = 0$, $D = E$, $\Pi' = {}^1/_2 (x_1^6 + 3x_1^2x_2^4 + 3x_1^4x_2^2 + x_2^6)$, $\omega = 1$, $V' = 0$ The corresponding Hamilton-Jacobi equation has a solution $S = {}^1/_4 (x_1^2 + x_2^2)^2$. It generates the system $x_1^{\cdot} = -x_2 + x_1(x_1^2 + x_2^2)$, $x_2^{\cdot} = x_1 + x_2(x_1^2 + x_2^2)$, whose singular point, as is readily verified, is a focus.

To prove the theorem, it will suffice to prove the existence of an analytic solution of the equation

$$ {}^1/_2\langle K^{-1}(\varepsilon x)(\partial S/\partial x - \varepsilon^{-1}V(\varepsilon x)), \partial S/\partial x - \varepsilon^{-1}V(\varepsilon x)\rangle + \varepsilon^{-2}\Pi(\varepsilon x) = 0 \tag{6} $$

in the region $G = \{x: |x_i| \leqslant 1, i = 1, 2\}$ for sufficiently small ε. To that end we use a well-known technique /2/. Define Banach spaces $A = (f, \|\cdot\|_1)$ and $B = (f, \|\cdot\|_2)$ of functions $f: R^2 \to R$ expressible in G as absolutely convergent power series

$$ f(x) = \sum_{\alpha_1+\alpha_2=2}^{\infty} a_{\alpha_1\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2}, \quad a_{\alpha_1\alpha_2} \in R $$

$$ \|f\|_1 = \sum_{\alpha_1+\alpha_2=2}^{\infty} (\alpha_1 + \alpha_2)|a_{\alpha_1\alpha_2}|, \quad \|f\|_2 = \sum_{\alpha_1+\alpha_2=2}^{\infty} |a_{\alpha_1\alpha_2}| $$

Write Eqs.(6) as a functional equation $F(S, \varepsilon) = 0$ and consider F as a map of some neighbourhood H of the point $(S_2, 0) \in A \times R$, where S_2 is defined as in (4), into B. The following assertions are true:

1) $S(S_2, 0) = 0$ and F is continuous at $(S_2, 0)$;
2) the derivative $F_s'(S, \varepsilon)$ of F exists in H and is continuous at $(S_2, 0)$;
3) $F_s'(S_2, 0) = \langle (A - \Omega)x, \partial/\partial x\rangle$.

We claim that the operator $F_s'(S_2, 0)$ has a bounded inverse. We may assume that this linear operator has been reduced to "canonical" form

$$ F_s'(S_2, 0) = \begin{cases} k_1x_1\partial/\partial x_1 + k_2x_2\partial/\partial x_2, & a_- > |\omega| \\ (ex_1 - gx_2)\partial/\partial x_1 + (gx_1 + ex_2)\partial/\partial x_2, & a_- < |\omega| \\ ex_1\partial/\partial x_1 + (ex_2 + \nu x_1)\partial/\partial x_2, & a_- = |\omega| \end{cases} $$

$$ g = \sqrt{|a_-^2 - \omega^2|}, \quad k_1 = e + g, \quad k_2 = e - g $$

Consider, e.g., the case $|a_-| < |\omega|$. Let $v(x) \in B$. The equation $F_s'(S_2, 0)u = v$ has a solution

$$ u = \sum_{k=2}^{\infty} u_k, \quad u_k = \sum_{\alpha_1+\alpha_2=k} u_{\alpha_1\alpha_2} x_1^{\alpha_1} x_2^{\alpha_2} $$

in which the coefficients of the form u_k are determined from the equation

$$ (E_{(k)} + g(ek)^{-1}C_{(k)})U_{(k)} = V_{(k)} \tag{7} $$

where $E_{(k)}$ is the $[(k + 1) \times (k + 1)]$ identity matrix, $C_{(k)}$ is the $[(k + 1) \times (k + 1)]$ matrix with elements $c_{i,i-1} = i - k - 1$ $(i = 1, \ldots, k)$, $c_{i,i+1} = i + 1$, its other elements vanishing; $U_{(k)}$ and $V_{(k)}$ are $[(k + 1) \times 1]$ matrices whose elements are the coefficients of the form u_k and v_k From (7) we obtain the inequality

$$ \gamma(U_{(k)}) \leqslant (ek)^{-1} \|(E_{(k)} + g(ek)^{-1}C_{(k)})^{-1}\| \gamma(V_{(k)}) $$

where $\|\cdot\|$ is the matrix norm induced by the vector norm $\gamma(X) = \sum_i |x_i|$. Since $\|E_{(k)}\| = 1$,

$\|C_{(k)}\| = k$ and $g/e < 1$, it follows that

$$k \sum_{\alpha_1+\alpha_2=k} |u_{\alpha_1\alpha_2}| \leqslant \frac{1}{e-g} \sum_{\alpha_1+\alpha_2=k} |v_{\alpha_1\alpha_2}|$$

whence we obtain $\|u\|_1 \leqslant c\|v\|_2$, $c = 1/(e+g)$.

The proof for the other cases is simpler.

Consequently, the operator $F_s'(S_2, 0)$ indeed has a bounded inverse.

By the Implicit Functions Theorem /4/, for small ε a unique solution $S(x, \varepsilon)$ of Eq.(6) exists, differing only slightly from $S_2(x)$. This proves the existence of an analytic solution S^+ of Eq.(1). Since the assumptions of the theorem are invariant when T_1 is replaced by $-T_1$ in the formula for the Lagrangian, this implies the existence of a second solution S.

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A REPRESENTATION OF THE SOLUTIONS OF THE GENERALIZED CAUCHY-RIEMANN EQUATIONS AND ITS APPLICATIONS*

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A method of integral representations for the generalized Cauchy-Riemann system in terms of an arbitrary analytic function, similar to the well-known Whittaker-Polozhii representation /1/, is developed. The representation includes various well-known results as special cases, and the limiting case leads to the classical representation of the theory of a generalized axisymmetric potential. The representations established are used to reduce mixed problems for the system to paired equations and then to a Fredholm equation of the second kind. At the same time, a device is described for regularizing paired equations, and a case in which a closed solution exists is presented.

The results are extended to a sytem of more-general form and also to second-order equations, whose type and dimensionality are not essential. It is shown that the integral operators constructed here convert the solution of a parabolic or hyperbolic equation with variable coefficients into a solution of the classical equations of heat conduction and wave propagation, thus furnishing an explicit representation for solutions of the corresponding Cauchy problems.

The effectiveness of the approach is demonstrated with reference to the problem of inflow in a fissure in an inhomogeneous layer of finite

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